

REGULARITY OF SOLUTIONS TO DEGENERATE p -LAPLACIAN EQUATIONS

DAVID CRUZ-URIBE, SFO, KABE MOEN, AND VIRGINIA NAIBO

ABSTRACT. We prove regularity results for solutions of the equation

$$\operatorname{div} \left(\langle AXu, Xu \rangle^{\frac{p-2}{2}} AXu \right) = 0,$$

$1 < p < \infty$, where $X = (X_1, \dots, X_m)$ is a family of vector fields satisfying Hörmander's ellipticity condition, A is an $m \times m$ symmetric matrix that satisfies degenerate ellipticity conditions. If the degeneracy is of the form

$$\lambda w(x)^{2/p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda w(x)^{2/p} |\xi|^2,$$

$w \in A_p$, then we show that solutions are locally Hölder continuous. If the degeneracy is of the form

$$k(x)^{\frac{-2}{p}} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq k(x)^{\frac{2}{p}} |\xi|^2,$$

$k \in A_{p'} \cap RH_\tau$, where τ depends on the homogeneous dimension, then the solutions are continuous almost everywhere, and we give examples to show that this is the best result possible. We give an application to maps of finite distortion.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the regularity of solutions of degenerate quasilinear elliptic equations that generalize the p -Laplacian

$$(1.1) \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

and the degenerate linear elliptic equations of the form

$$(1.2) \quad Lu = \operatorname{div}(A \nabla u) = 0,$$

where A is an $n \times n$ real symmetric matrix that satisfies the ellipticity condition

$$(1.3) \quad \lambda w(x) |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda w(x) |\xi|^2,$$

Date: April 24, 2012.

2000 Mathematics Subject Classification. Primary 42A50.

Key words and phrases. p -Laplacian, Hörmander vector fields, A_p weights.

The first author is supported by the Stewart-Dorwart Faculty Development Fund at Trinity College and grant MTM2009-08934 from the Spanish Ministry of Science and Innovation. The second author is partially supported by a summer research grant from the University of Alabama. The third author is partially supported by the NSF under grant DMS 1101327.

with w in the Muckenhoupt class A_2 . In the simplest case we prove the following results. Throughout this paper, Ω will denote an open, connected subset of \mathbb{R}^n . For precise definitions of the weight classes A_p and RH_t , see Section 2 below.

Theorem A. *Given Ω and p , $1 < p < \infty$, let A be an $n \times n$ real-symmetric matrix of measurable functions defined in Ω such that for every $\xi \in \mathbb{R}^n$ and $x \in \Omega$,*

$$(1.4) \quad \lambda w(x)^{2/p} |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq \Lambda w(x)^{2/p} |\xi|^2,$$

where $w \in A_p$ and $0 < \lambda < \Lambda < \infty$. Then every solution u of the equation

$$(1.5) \quad \mathcal{L}_p u(x) = \operatorname{div} \left(\langle A(x) \nabla u(x), \nabla u(x) \rangle^{\frac{p-2}{2}} A(x) \nabla u(x) \right) = 0$$

is Hölder continuous on compact subsets of Ω .

Theorem B. *Given Ω and p , $1 < p < \infty$, let A be an $n \times n$ real-symmetric matrix of measurable functions defined in Ω such that for every $\xi \in \mathbb{R}^n$ and $x \in \Omega$,*

$$(1.6) \quad k(x)^{\frac{-2}{p'}} |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq k(x)^{\frac{2}{p}} |\xi|^2,$$

where $k \in A_{p'} \cap RH_n$. Then every solution u of (1.5) is continuous almost everywhere in Ω .

Continuity almost everywhere is the best result possible with these hypotheses. To illustrate this, consider the following example adapted from [6]. Let $p = 2$, let Ω be the unit ball in \mathbb{R}^2 and fix $q > 2$. Define $k(x, y) = |x|^{-1/q}$. Then $k \in A_1 \cap RH_2$. Define the matrix A by

$$A(x) = \begin{pmatrix} k(x)^{-1} & 0 \\ 0 & k(x) \end{pmatrix};$$

then A satisfies the ellipticity condition (5.1). The function

$$u(x, y) = \frac{x}{|x|} \exp(|x|^{1/q'}) \sin(y/q')$$

is a (formal) solution of (1.5) but it is discontinuous on the y -axis. As we will see below in Section 4, our results actually give us precise control on the set of discontinuities of u .

To put these theorems in context, we describe some known results. It is well-known that solutions to the p -Laplacian (1.1) are in $C_{loc}^{1,\alpha}$ for some α , $0 < \alpha < 1$ (see [9, 24, 33]). Fabes, Kenig and Serapioni [10] showed that given (1.3), the solutions of (1.2) are Hölder continuous. Modica [29], working in the setting of variational integrals, extended their results to non-linear equations and proved a version of Theorem A.

In the more general case when the matrix A satisfies the two-weight ellipticity condition

$$w(x) |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq v(x) |\xi|^2,$$

where $w(x) \leq v(x)$ are non-negative measurable functions, then in general solutions of (1.2) will not be regular. However, Chanillo and Wheeden [6] proved that if v, w are doubling, $w \in A_2$, and there exists $q > 2$ such that given two balls $B(x_1, r_1) \subset B(x_2, r_2)$,

$$(1.7) \quad \frac{r_1}{r_2} \left(\frac{v(B(x_1, r_1))}{v(B(x_2, r_2))} \right)^{1/q} \leq C \left(\frac{w(B(x_1, r_1))}{w(B(x_2, r_2))} \right)^{1/2},$$

then solutions of (1.2) satisfy a Harnack inequality. As a consequence, they showed that if we define

$$\mu(x, r) = \left(\frac{v(B(x, r))}{w(B(x, r))} \right)^{1/2},$$

then the solution is continuous provided that

$$\mu(x, r) = o(\log \log(r^{-1}))$$

as $r \rightarrow 0$. Ferrari [11] extended these results to non-negative solutions of the degenerate p -Laplacian (1.5) when the matrix A satisfies

$$w(x)^{2/p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq v(x)^{2/p} |\xi|^2$$

assuming v, w are doubling, $w \in A_p$, $q > p$ and (1.7) holds with the exponent $1/2$ replaced by $1/p$.

Using the results of Chanillo and Wheeden, Theorem B in the case $p = 2$ was recently proved by the first author, di Gironimo and Sbordonc [7]. Degenerate elliptic equations of this type arise in the study of maps of finite distortion in the plane: see, for example, [1].

We will actually prove our results in the more general setting of C^∞ vector fields. This introduces a different kind of degeneracy: while the matrix A can be thought of as making the measure degenerate, the vector fields introduces degeneracies into the geometry. Let X_1, \dots, X_m be a family of C^∞ vector fields on \mathbb{R}^n that satisfy Hörmander's hypoellipticity condition. Define the operators $Xu = (X_1 u, \dots, X_m u)$ and

$$\operatorname{div}_X \mathbf{v} = - \sum_{j=1}^m X_j^* v_j,$$

where $\mathbf{v} = (v_1, \dots, v_m)$. Then the operators above have natural generalizations to Carnot-Carathéodory spaces, and in particular to Carnot groups. Capogna, Danielli and Garofalo [4] showed that solutions of

$$\operatorname{div}_X (|Xu|^{p-2} Xu) = 0$$

are Hölder continuous. Lu [27] proved the regularity of solutions of

$$\operatorname{div}_X (AXu) = 0$$

where A is an $m \times m$ real-symmetric matrix that satisfies (1.3) for $w \in A_2$. Franchi, Lu and Wheeden [12, 13] extended the results of Chanillo and Wheeden [6] to this more general setting. And Ferrari [11] in turn extended these results to the solutions of the equation

$$\operatorname{div}_X(\langle AXu, Xu \rangle^{\frac{p-2}{2}} AXu) = 0.$$

The main results of this paper extend Theorems A and B to the setting of Carnot-Carathéodory spaces. Theorem 4.1 generalizes Theorem A, and Theorems 4.4 and 4.5 generalize Theorem B. Since the statement of all of these results requires some additional definitions, we defer their precise statement until below.

Our results are applicable to mappings of finite distortion. A function $f : \Omega \rightarrow \mathbb{R}^n$ with $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$ and locally integrable Jacobian is a mapping of finite distortion if there is a function $K \geq 1$ finite a.e. such that

$$|Df(x)|^n \leq K(x)J_f(x),$$

where $|Df(x)| = \sup_{|h|=1} |Df(x)h|$ is the operator norm and $J_f(x) = \det Df(x)$. These mappings have been extensively studied: see Astala, Iwaniec, and Martin [1, Chapter 20] for mappings in the plane and Iwaniec and Martin [20, Chapters 6 and 7] for results in higher dimensions. The regularity of such mappings has been extensively studied. Mappings of finite distortion satisfy an equation of the form (1.5) with the degeneracy controlled by the distortion function. As an application of our results we show that with assumptions on the distortion, mappings of finite distortion are continuous almost everywhere. This complements work of Manfredi [28].

The remainder of this paper is organized as follows. In Section 2 we gather some preliminary results. In §2.1 we recall the definition and basic properties of Hörmander vector fields, including Carnot groups as a particular case; in §2.2 we discuss the theory of Muckenhoupt weights in the context of spaces of homogeneous type. In Section 3 we give the machinery we need to work with the equations we are interested in. In §3.1- 3.3 we define precisely the solution to $\mathcal{L}_p u = 0$. We follow the definition given in [11]; for the convenience of the reader we provide full details. In §3.4 we give the Harnack inequalities we use in our main proofs. In Section 4 we prove Theorems 4.1, 4.4 and 4.5. In Section 5 we show that Theorem B is sharp: we construct an operator \mathcal{L}_p and a solution u of $\mathcal{L}_p u = 0$ that is discontinuous on a set of measure zero. Finally, in Section 6 we give our applications to the theory of mappings of finite distortion.

Throughout this paper, C , c , etc. will denote a constant whose values may change from line to line. Generally, it will depend on the value of p , the dimension of the vector field and the characteristics of the weight, but not on the specific function.

Acknowledgement. The authors completed this collaboration while at the American Institute of Mathematics. They would like to thank AIM for the hospitality and support.

2. PRELIMINARY RESULTS

2.1. Hörmander vector fields. Let $X = (X_1, \dots, X_m)$ be a family of infinitely differentiable vector fields defined in \mathbb{R}^n and with values in \mathbb{R}^n . We identify each X_j with the first order differential operator acting on Lipschitz functions given by

$$X_j f(x) = X_j(x) \cdot \nabla f(x), \quad j = 1, \dots, m.$$

We define $Xf = (X_1 f, X_2 f, \dots, X_m f)$ and set

$$|Xf(x)| = \left(\sum_{j=1}^m |X_j f(x)|^2 \right)^{\frac{1}{2}}.$$

Suppose that X satisfies Hörmander's condition in \mathbb{R}^n : that is, there exists $l \in \mathbb{N}$ such that the family of commutators of vector fields in X up to length l span \mathbb{R}^n at every point in \mathbb{R}^n . Let C_X be the family of absolutely continuous curves $\zeta : [a, b] \rightarrow \mathbb{R}^n$, $a \leq b$, such that there exist measurable functions $c_j(t)$, $a \leq t \leq b$, $j = 1, \dots, m$, satisfying $\sum_{j=1}^m c_j(t)^2 \leq 1$ and $\zeta'(t) = \sum_{j=1}^m c_j(t) X_j(\zeta(t))$ for almost every $t \in [a, b]$. Given $x, y \in \mathbb{R}^n$, define

$$\rho(x, y) = \inf \{T > 0 : \exists \zeta \in C_X \text{ s.t. } \zeta(0) = x, \zeta(T) = y\}.$$

The function ρ is a metric on \mathbb{R}^n called the Carnot-Carathéodory metric associated to X , and the pair (\mathbb{R}^n, ρ) is said to be a Carnot-Carathéodory space. We note that all of the above definitions can be made when \mathbb{R}^n is replaced by an open connected subset of \mathbb{R}^n ; we refer the reader to [31] for more details on Carnot-Carathéodory spaces.

Hereafter, $X = (X_1, \dots, X_m)$ will be a family of C^∞ vector fields defined in \mathbb{R}^n that satisfies Hörmander's condition in \mathbb{R}^n , ρ will denote the corresponding Carnot-Carathéodory metric in \mathbb{R}^n and $Q \geq n$ will indicate the homogeneous dimension of \mathbb{R}^n with respect to X . Also, by a ball B of radius r and center x we mean the metric ball $B = B(x, r) = \{y \in \mathbb{R}^n : \rho(x, y) < r\}$. Given a ball B we will let $r(B)$ denote its radius. Nagel, Stein and Wainger proved in [31] that for every compact set $K \subset \mathbb{R}^n$ there exist positive constants R_0 , c , c_1 and c_2 depending on K such that

$$(2.1) \quad |B(x, 2r)| \leq c |B(x, r)|, \quad x \in K, 0 < r < R_0$$

and

$$(2.2) \quad c_1 |x - y| \leq \rho(x, y) \leq c_2 |x - y|^{\frac{1}{t}}, \quad x, y \in K.$$

Note that inequality (2.2) implies that the topology induced by ρ on \mathbb{R}^n is the same as the Euclidean topology. Using (2.1), Bramanti and Brandolini proved in [3, p. 533] that the triple (B, ρ, dx) is a space of homogeneous type for all metric balls $B \subset \mathbb{R}^n$; moreover, structural constants are uniform for all metric balls with radius bounded by a fixed constant and with center in a fixed compact subset of \mathbb{R}^n . In addition, they gave regularity properties for ∂U , $U \subset \mathbb{R}^n$, that imply that (U, ρ, dx) is a space of homogeneous type. One such condition, possessed by every metric ball [3, p. 533], is

$$(2.3) \quad |B(x, r) \cap U| \geq C |B(x, r)|, \quad x \in U, 0 < r < \text{diam}_\rho(U).$$

In the particular case when X is the generator of a homogeneous Carnot group, $|B(x, r)| = |B(0, 1)| r^Q$ and therefore (\mathbb{R}^n, ρ, dx) is a space of homogeneous type. For a detailed description of the theory of Carnot groups see, for instance, the book by Bonfiglioli, Lanconelli and Uguzzoni [2].

The following upper and lower estimates play a significant role in our proofs of Theorems 4.4 and 4.5.

Lemma 2.1. *Given any compact set $K \subset \mathbb{R}^n$, there exist positive constants d_K, D_K, R_K such that given $x_1, x_2 \in K$ and $0 < r_1 < r_2 < R_K$, if $B(x_1, r_1) \subset B(x_2, r_2)$, then*

$$(2.4) \quad d_K \left(\frac{r_1}{r_2} \right)^Q \leq \frac{|B(x_1, r_1)|}{|B(x_2, r_2)|} \leq D_K \left(\frac{r_1}{r_2} \right)^n.$$

Proof. Fix K ; when the balls are concentric, there exist constants c_K, C_K and r_K such that

$$(2.5) \quad c_K \left(\frac{s}{r} \right)^Q \leq \frac{|B(x, s)|}{|B(x, r)|} \leq C_K \left(\frac{s}{r} \right)^n$$

whenever $x \in K$ and $0 < s < r < r_K$. The proof of (2.5) can be found in [12, p. 584]. Let $R_K = r_K/4$. Fix balls $B(x_1, r_1)$ and $B(x_2, r_2)$ as in the hypotheses. Then $B(x_2, r_2) \subset B(x_1, 2r_2)$, and so by the first inequality in (2.5),

$$\frac{|B(x_1, r_1)|}{|B(x_2, r_2)|} \geq \frac{|B(x_1, r_1)|}{|B(x_1, 2r_2)|} \geq c_K \left(\frac{r_1}{2r_2} \right)^Q,$$

so we get the first inequality in (2.4) with $d_K = 2^{-Q} c_K$.

Similarly, we have that $B(x_2, r_2) \subset B(x_1, 2r_2) \subset B(x_2, 4r_2)$, and so again by the first inequality in (2.5),

$$4^Q c_K^{-1} |B(x_2, r_2)| \geq |B(x_2, 4r_2)| \geq |B(x_1, 2r_2)|.$$

Hence by the second inequality in (2.5),

$$\frac{|B(x_1, r_1)|}{|B(x_2, r_2)|} \leq 4^Q c_K^{-1} \frac{|B(x_1, r_1)|}{|B(x_1, 2r_2)|} \leq 4^Q c_K^{-1} C_K \left(\frac{r_1}{2r_2} \right)^n.$$

Therefore, we get the second inequality in (2.4) with $D_K = 2^{2Q-n} c_K^{-1} C_K$. \square

2.2. Muckenhoupt weights in spaces of homogenous type. The theory of Muckenhoupt weights in spaces of homogenous type is well-developed and parallels the standard theory.

By a weight we will mean a positive function in $L_{loc}^1(\mathbb{R}^n)$. We will say that a weight w is doubling in $U \subset \mathbb{R}^n$ if there exists a constant C_U such that

$$w(B(x, 2r) \cap U) \leq C_U w(B(x, r) \cap U), \quad x \in U, r > 0.$$

We will say that w is locally doubling in U if for each compact set $K \subset U$ and $R > 0$ there exists $C_{K,R}$ such that

$$w(B(x, 2r) \cap U) \leq C_{K,R} w(B(x, r) \cap U), \quad x \in K, 0 < r < R.$$

Given p , $1 < p < \infty$, and a weight w , we say $w \in A_p(U, \rho, dx)$ if

$$[w]_{A_p} = \sup_{\substack{r>0 \\ x \in U}} \left(\int_{B(x,r) \cap U} w(x) dx \right) \left(\int_{B(x,r) \cap U} w(x)^{1-p'} dx \right)^{p-1} < \infty.$$

When $p = 1$, we say $w \in A_1(U, \rho, dx)$ if

$$[w]_{A_1} = \operatorname{ess\,sup}_{x \in U} \frac{Mw(x)}{w(x)} < \infty,$$

where M is the maximal operator defined using metric balls: given a measurable function f , for all $x \in U$,

$$Mf(x) = \sup_{r>0} \int_{B(x,r) \cap U} |f(y)| dy.$$

Finally, we set $A_\infty(U, \rho, dx) := \cup_{p=1}^\infty A_p(U, \rho, dx)$.

We say that a weight w satisfies a reverse Hölder inequality in U with exponent $t > 1$, denoted by $w \in RH_t(U, \rho, dx)$, if

$$[w]_{RH_t} = \sup_{\substack{r>0 \\ x \in U}} \frac{\left(\int_{B(x,r) \cap U} w(x)^t dx \right)^{1/t}}{\int_{B(x,r) \cap U} w(x) dx} < \infty.$$

Remark 2.2. If U is a bounded set that satisfies (2.3) then $|B| \sim |B \cap U|$ for any metric ball with center in U and $r(B) \leq \operatorname{diam}_\rho(U)$. Therefore, if $w \in A_p(\mathbb{R}^n, \rho, dx)$, then the restriction of w to U belongs to $A_p(U, \rho, dx)$. In particular, this is true when U is a metric ball.

Muckenhoupt weights have the following properties.

Lemma 2.3. *Let $U \subset \mathbb{R}^n$ and suppose that (U, ρ, dx) is a space of homogeneous type. If $w \in A_p(U, \rho, dx)$ for some $p > 1$, then the following are true:*

- (a) $w^{1-p'} \in A_{p'}(U, \rho, dx)$;
- (b) *there exists $t > 1$ such that $w \in RH_t(U, \rho, dx)$;*
- (c) *there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}(U, \rho, dx)$.*

As in the Euclidean case, part (a) is a direct consequence of the definition of A_p weight and part (c) easily follows from (a) and (b).

Lemma 2.4. *Let $U \subset \mathbb{R}^n$.*

- (a) *If $w \in RH_t(U, \rho, dx)$ for some $t > 1$, then for any $x \in U$, $r > 0$, and any measurable set $E \subset B(x, r) \cap U$,*

$$\frac{w(E)}{w(B(x, r) \cap U)} \leq [w]_{RH_t(U, \rho, dx)} \left(\frac{|E|}{|B(x, r) \cap U|} \right)^{1/t'}.$$

(b) If $w \in A_p(U, \rho, dx)$ for some $p > 1$, then for any $x \in U$, $r > 0$, and any measurable set $E \subset B(x, r) \cap U$,

$$\frac{|E|}{|B(x, r) \cap U|} \leq \left([w]_{A_p(U, \rho, dx)} \frac{w(E)}{w(B(x, r) \cap U)} \right)^{1/p}.$$

In particular, if (U, ρ, dx) is a space of homogeneous type, then w is doubling: for all $x \in U$ and $r > 0$,

$$w(B(x, 2r) \cap U) \leq D^p [w]_{A_p(U, \rho, dx)} w(B(x, r) \cap U),$$

where D is a doubling constant for (U, ρ, dx) .

In the classical case these estimates can be found in [14]. The proofs in the setting of spaces of homogeneous type are essentially the same, replacing Euclidean balls/cubes with metric balls. The doubling property is a consequence of the first inequality in part (b) and the fact that (U, ρ, dx) is a space of homogenous type.

Remark 2.5. In Lemma 2.3, we note that $[w^{1-p'}]_{A_{p'}(U, \rho, dx)} = [w]_{A_p(U, \rho, dx)}^{1/(p-1)}$ and the constants t , $[w]_{RH_t(U, \rho, dx)}$, ϵ , and $[w]_{A_{p-\epsilon}(U, \rho, dx)}$ depend only on p , bounds for $[w]_{A_p(U, \rho, dx)}$ and structural constants of (U, ρ, dx) . In particular, if $w \in A_p(\mathbb{R}^n, \rho, dx)$, all these constants are uniformly bounded on U when U is a metric ball with radius bounded by a fixed constant and center in a fixed compact subset $K \subset \mathbb{R}^n$: in this case the constant C in (2.3) can be taken independent of U . Similarly, all constants appearing in the inequalities in Lemma 2.4 are uniformly bounded on U , when $w \in A_p(\mathbb{R}^n, \rho, dx)$ and U is a metric ball with radius bounded by a fixed constant and center in a fixed compact subset $K \subset \mathbb{R}^n$.

Remark 2.6. Note that if $w \in A_p(\Omega, \rho, dx)$ then w is locally doubling. This follows from the last inequality in Lemma 2.4 and Remark 2.5.

Lemma 2.7. Let $U \subset \mathbb{R}^n$ and suppose that (U, ρ, dx) is a space of homogeneous type. If $w \in A_p(U, \rho, dx) \cap RH_t(U, \rho, dx)$, then $w^t \in A_q(U, \rho, dx)$, where $q = t(p-1) + 1$.

This result is proved in the classical setting in [21]. The proof is the same in the setting of spaces of homogeneous type.

3. THE MACHINERY OF PDES

Though we are ultimately interested in degenerate equations where the degeneracy has the particular form of (1.4) and (1.6), we need to work in a more general setting. Throughout this section, $\Omega \subset \mathbb{R}^n$ is open and connected, $X = (X_1, \dots, X_m)$ is a family of C^∞ vector fields defined in \mathbb{R}^n and satisfying Hörmander's condition in \mathbb{R}^n , ρ is the corresponding Carnot-Carathéodory metric in \mathbb{R}^n and Q is the homogeneous dimension of \mathbb{R}^n with respect to X . By $\text{Lip}(\overline{\Omega})$ we mean the collection of Lipschitz functions on $\overline{\Omega}$ and by $\text{Lip}_0(\Omega)$ the collection of Lipschitz functions whose support is compactly contained in Ω .

3.1. Admissible weights. We begin with a definition of the general class of weights we will be considering.

Definition 3.1. Given a pair of weights (w, v) , $w \leq v$, $v, w \in L^1_{\text{loc}}(\mathbb{R}^n)$, and p , $1 < p < \infty$, we say that the pair is p -admissible in Ω if:

- (1) v is locally doubling in (Ω, ρ, dx) and $w \in A_p(\Omega, \rho, dx)$.
- (2) Given a compact set $K \subset \Omega$, there exists $q > p$ and $r_0 > 0$ such that if B is a ball with center in K and $r(B) < r_0$, then $B \subset \Omega$ and

$$\frac{r(B_1)}{r(B_2)} \left(\frac{v(B_1)}{v(B_2)} \right)^{1/q} \leq C \left(\frac{w(B_1)}{w(B_2)} \right)^{1/p}$$

for all balls $B_1 \subseteq B_2 \subseteq B$.

Remark 3.2. Our definition generalizes to the two-weight case the definition of p -admissible weight given in [17].

The importance of this class is that these conditions imply a two-weight Poincaré inequality. When $X = \nabla$, the following result is due to Chanillo and Wheeden [5]; in the general case it is due to Franchi, Lu and Wheeden [12].

Theorem 3.3. *Given p , $1 < p < \infty$, a pair of p -admissible weights (w, v) in Ω , and a compact set $K \subset \Omega$, there exists $r_0 > 0$ (depending on K and X) such that if B is a ball with center in K and $r(B) < r_0$, then for all $f \in \text{Lip}(\overline{B})$,*

$$(3.1) \quad \left(\frac{1}{v(B)} \int_B |f - f_{B,v}|^q v \, dx \right)^{1/q} \leq Cr \left(\frac{1}{w(B)} \int_B |Xf|^p w \, dx \right)^{1/p},$$

where $f_{B,v} = \frac{1}{v(B)} \int_B f(y) v(y) \, dy$.

We will need two corollaries of Theorem 3.3. The first, Corollary 3.4, is a local Sobolev inequality. The proof follows from a similar line of reasoning to that in [16, p. 80, Theorem 13.1].

Corollary 3.4. *With the same hypotheses as Theorem 3.3, there exists $R_0 > 0$ such that if B is a ball with center in K and $r(B) < R_0$, then for all $f \in \text{Lip}_0(B)$,*

$$(3.2) \quad \left(\frac{1}{v(B)} \int_B |f|^q v \, dx \right)^{1/q} \leq Cr \left(\frac{1}{w(B)} \int_B |Xf|^p w \, dx \right)^{1/p}.$$

Below, we will need to assume a Sobolev inequality on Ω : for all $f \in \text{Lip}_0(\Omega)$,

$$(3.3) \quad \left(\int_{\Omega} |f(x)|^p v(x) \, dx \right)^{1/p} \leq C \left(\int_{\Omega} |Xf(x)|^p w(x) \, dx \right)^{1/p}.$$

In the particular case when $X = \nabla$, there are no restrictions on the size of the balls for which the Poincaré inequality holds, and so (3.3) holds for any bounded Ω . In the

general case, the “global” Sobolev inequality (3.3) can be proved to be a consequence of Corollary 3.4 when Ω is such that a Hardy-type inequality is satisfied.

Corollary 3.5. *Let $1 < p < \infty$ and consider a pair of p -admissible weights (w, v) in a neighborhood of Ω . If Ω is a bounded domain such that*

$$(3.4) \quad \int_{\Omega} \frac{|f|^p}{\text{dist}_{\rho}(x, \partial\Omega)^p} w \, dx \leq C_{\Omega} \int_{\Omega} |X(f)|^p w \, dx, \quad f \in \text{Lip}_0(\Omega),$$

then (3.3) holds for $f \in \text{Lip}_0(\Omega)$.

We omit the proof of Corollary 3.5 because it is in the spirit of the proof of [8, Theorem 4.3]. It requires a Whitney decomposition for Ω in terms of metric balls, and a corresponding partition of unity.

3.2. The spaces $S^p(\Omega)$ and $S_0^p(\Omega)$. We now define the spaces in which weak solutions to our equations live. For simplicity, we state our definition assuming that Ω is bounded. If $\Omega = \mathbb{R}^n$, then we have to modify our definition by everywhere replacing $\text{Lip}(\overline{\Omega})$ with $\text{Lip}_0(\mathbb{R}^n)$.

In this section we assume that (w, v) is a pair of p -admissible weights and $A = A(x)$ is a real, locally integrable, $m \times m$ symmetric matrix defined in Ω that satisfies the degenerate ellipticity condition

$$(3.5) \quad w(x)^{2/p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq v(x)^{2/p} |\xi|^2 \quad \xi \in \mathbb{R}^m, x \in \Omega.$$

We define the space $L_A^p(\Omega)$ to consist of all measurable vector valued functions \mathbf{f} such that

$$\|\mathbf{f}\|_A = \left(\int_{\Omega} \langle A\mathbf{f}, \mathbf{f} \rangle^{p/2} dx \right)^{1/p} < \infty.$$

Since A is symmetric and positive semi-definite, $A^{1/2}$ exists; hence,

$$\langle A\mathbf{f}, \mathbf{f} \rangle^{1/2} = |A^{1/2}\mathbf{f}|.$$

It follows immediately from this identity that $\|\cdot\|_A$ is a norm. In [32] (see also [30]) it was shown that with this norm $L_A^p(\Omega)$ is a Banach space of equivalence classes of functions such that $\|\mathbf{f} - \mathbf{g}\|_A = 0$. However, by the ellipticity condition, we have that

$$\|\mathbf{f}\|_A \geq \left(\int_{\Omega} |\mathbf{f}|^p w \, dx \right)^{1/p};$$

since $w \in A_p$ it is non-zero almost everywhere, so $\|\mathbf{f}\|_A = 0$ if and only if $\mathbf{f} = \mathbf{0}$ almost everywhere. Hence, we can identify the elements of $L_A^p(\Omega)$ as functions defined up to a set of measure 0.

If Ω is bounded and $u \in \text{Lip}(\overline{\Omega})$, then $Xu \in L_A^p(\Omega)$: by the ellipticity condition and since $v \in L_{loc}^1(\mathbb{R}^n)$,

$$\|Xu\|_A \leq \left(\int_{\Omega} |Xu|^p v \, dx \right)^{1/p} \leq \|Xu\|_{\infty} v(\Omega)^{1/p} < \infty.$$

Therefore, we can define the space $S^p(\Omega)$ to be the closure of $\text{Lip}(\overline{\Omega})$ with respect to the norm

$$\|u\| = \|u\|_{L^p(v, \Omega)} + \|Xu\|_A.$$

Properly, $S^p(\Omega)$ is a Banach space that consists of equivalence classes of Cauchy sequences $\{u_j\}$ of Lipschitz functions. However, this sequence converges to a function $u \in L^p(v, \Omega)$, and the sequence $\{Xu_j\}$ converges to a function $\mathbf{U} \in L_A^p(\Omega)$ and the map $\{u_j\} \mapsto (u, \mathbf{U})$ is an isomorphism between $S^p(\Omega)$ and a closed subset of $L^p(v, \Omega) \times L_A^p(\Omega)$.

Furthermore, if u is the first element of such a pair, then \mathbf{U} is uniquely determined. Suppose to the contrary that (u, \mathbf{U}) and (u, \mathbf{V}) are in the image. This is equivalent to saying that there exists a Cauchy sequence $\{u_j\}$ in $S^p(\Omega)$ such that $u_j \rightarrow 0$ in $L^p(v, \Omega)$ and $Xu_j \rightarrow \mathbf{U} \neq 0$ in $L_A^p(\Omega)$. By the ellipticity conditions and since $w \leq v$, the same limits hold in $L^p(w, \Omega)$. However, since $w \in A_p(\Omega, \rho, dx)$, this is impossible. Let $B \subset \Omega$ be any ball and let $g \in C_0^\infty(B)$. Then

$$\begin{aligned} \left| \int_B g \mathbf{U} \, dx \right| &\leq \left| \int_B g(\mathbf{U} - Xu_j) \, dx \right| + \left| \int_B g Xu_j \, dx \right| \\ &\leq \left(\int_B |\mathbf{U} - Xu_j|^p w \, dx \right)^{1/p} \left(\int_B |g|^{p'} w^{1-p'} \, dx \right)^{1/p'} \\ &\quad + \left(\int_B |u_j|^p w \, dx \right)^{1/p} \left(\int_B |Xg|^{p'} w^{1-p'} \, dx \right)^{1/p}. \end{aligned}$$

By assumption, the first integral in each term goes to 0; since $w^{1-p'} \in A_{p'}(\Omega, \rho, dx)$, and $g, Xg \in L^\infty$, the second integral in each term is finite. Therefore, taking the limit, we get that the initial integral is 0; since this holds for all such B and g , $\mathbf{U} = 0$.

Remark 3.6. The uniqueness of this representation of a Cauchy sequence $\{u_j\}$ by the pair (u, \mathbf{U}) depends strongly on the fact that $w \in A_p(\Omega, \rho, dx)$. In general, this need not be the case, as is shown by the example in [10, p. 91].)

Given (u, \mathbf{U}) as above with $\mathbf{U} = (U_1, \dots, U_m)$, then u is a distribution and \mathbf{U} corresponds to the distributional derivative of u : that is,

$$(3.6) \quad \int_{\Omega} U_i \varphi \, dx = \int_{\Omega} u X_i^* \varphi \, dx \quad \varphi \in C_0^\infty(\Omega).$$

This follows from a calculation similar to that used to prove the uniqueness of \mathbf{U} . Indeed, we first note that both of the integrals appearing in (3.6) exist: after multiplying and dividing by w , use Hölder's inequality, and that $u, \mathbf{U} \in L^p(w, \Omega)$, φ and $X_i^* \varphi$ are bounded, and $w^{1-p'} \in L^1_{\text{loc}}(\Omega)$. To prove the equality, let $\{u_j\}$ be a Cauchy sequence of Lipschitz functions in $S^p(\Omega)$ associated to (u, \mathbf{U}) . Then

$$\int_{\Omega} |u_j - u|^p w \, dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} |\mathbf{U} - Xu_j|^p w \, dx \rightarrow 0,$$

and if $\varphi \in C_0^\infty(\Omega)$ with $\text{supp}(\varphi) = D$,

$$\begin{aligned} \left| \int_{\Omega} u X_i^* \varphi - U_i \varphi \, dx \right| &= \left| \int_D (u - u_j) X_i^* \varphi + \int_D (U_i - Xu_j) \varphi \, dx \right| \\ &\leq \|X_i^* \varphi\|_{\infty} w^{1-p'}(D) \left(\int_D |u - u_j|^p w \, dx \right)^{1/p} \\ &\quad + w^{1-p'}(D) \|\varphi\|_{\infty} \left(\int_D |U_i - Xu_j|^p w \, dx \right)^{1/p}, \end{aligned}$$

and the last terms go to zero as $j \rightarrow \infty$.

Hereafter, given a pair (u, \mathbf{U}) representing a Cauchy sequence of Lipschitz functions in $S^p(\Omega)$, we use Xu instead of \mathbf{U} and, by a small abuse of notation, we write $u \in S^p(\Omega)$.

We can also see that $S^p(\Omega)$ is a reflexive Banach space, a fact we will need below. Given $(u, \mathbf{U}) \in L^p(v, \Omega) \times L_A^p(\Omega)$, the map $T(u, \mathbf{U}) = (u, A^{1/2} \mathbf{U})$ is an isometry into $L^p(v, \Omega) \times L^p(\Omega)$ which is a reflexive space since $1 < p < \infty$. Identifying $S^p(\Omega)$ with its image, we have that $T(S^p(\Omega))$ is a closed subspace of a reflexive Banach space; it follows that $S^p(\Omega)$ is also reflexive.

We now define the space $S_0^p(\Omega)$ to be the closure of $\text{Lip}_0(\Omega)$ in $L_A^p(\Omega)$: more precisely, $S_0^p(\Omega)$ consists of the equivalence classes of sequences $\{u_j\}$ such that $\{Xu_j\}$ converges in $L_A^p(\Omega)$. By an abuse of notation, we will denote the Cauchy sequences with a function u . If the “global” Sobolev inequality (3.3) holds, then $S_0^p(\Omega) \subset S^p(\Omega)$ and so this identification can be made precise. To see this inclusion, fix a Cauchy sequence $\{u_j\}$ in $S_0^p(\Omega)$. Then this sequence is Cauchy in $L^p(v, \Omega)$: by (3.3) and the ellipticity condition,

$$\left(\int_{\Omega} |u_j - u_k|^p v \, dx \right)^{1/p} \leq C \left(\int_{\Omega} |X(u_j - u_k)|^p w \, dx \right)^{1/p} \leq C \|X(u_j - u_k)\|_A.$$

The limit gives us the desired function u .

Remark 3.7. We note that condition (2) of Definition 3.1 is not used in this section except to obtain the embedding $S_0^p(\Omega) \subset S^p(\Omega)$ via Corollary 3.5.

3.3. Weak solutions. In this section we again assume that (w, v) is a pair of p -admissible weights in Ω and let $A = A(x)$ be an $m \times m$ real symmetric matrix defined in Ω that satisfies the degenerate ellipticity condition (3.5). Remark 3.7 also applies to this section.

To define weak solutions, we first need to define a functional that acts in some ways like an inner product on $S^p(\Omega)$. Given $u, \varphi \in \text{Lip}(\overline{\Omega})$, define the functional

$$a_0^p(u, \varphi) = \int_{\Omega} \langle AXu, Xu \rangle^{\frac{p-2}{2}} \langle AXu, X\varphi \rangle dx.$$

This quantity is finite for all such u and φ . Since A is symmetric we have the inequality

$$|\langle A\xi, \eta \rangle| \leq \langle A\xi, \xi \rangle^{1/2} \langle A\eta, \eta \rangle^{1/2}.$$

Hence, by Hölder's inequality,

$$\begin{aligned} |a_0^p(u, \varphi)| &\leq \int_{\Omega} \langle AXu, Xu \rangle^{\frac{p-1}{2}} \langle AX\varphi, X\varphi \rangle^{1/2} dx \\ &\leq \left(\int_{\Omega} \langle AX\varphi, X\varphi \rangle^{p/2} dx \right)^{1/p} \left(\int_{\Omega} \langle AXu, Xu \rangle^{p/2} dx \right)^{1/p'} \\ (3.7) \quad &= \|X\varphi\|_A \|Xu\|_A^{p-1}. \end{aligned}$$

We can extend a_0^p to functions $u, \varphi \in S^p(\Omega)$ or even in $S_0^p(\Omega)$.

Lemma 3.8. *If $u, \varphi \in S^p(\Omega)$ are the limits of the Cauchy sequences $\{u_k\}_{k \in \mathbb{N}} \subset \text{Lip}(\overline{\Omega})$ and $\{\varphi_k\}_{k \in \mathbb{N}} \subset \text{Lip}(\overline{\Omega})$, then*

$$a_0^p(u, \varphi) := \lim_{k \rightarrow \infty} a_0^p(u_k, \varphi_k) = \lim_{k \rightarrow \infty} \int_{\Omega} \langle AXu_k, Xu_k \rangle^{\frac{p-2}{2}} \langle AXu_k, X\varphi_k \rangle dx$$

is well defined: the limit exists and does not depend on the choice of Cauchy sequences. The same conclusion holds if u or φ are in $S_0^p(\Omega)$.

Proof. We prove this for $u, \varphi \in S^p(\Omega)$; the proof for $S_0^p(\Omega)$ is identical. Fix Cauchy sequences $\{u_k\}$ and $\{\varphi_k\}$ in $\text{Lip}(\overline{\Omega})$; we will prove that $\{a_0^p(u_k, \varphi_k)\}$ is a Cauchy sequence in \mathbb{R} . By the definition,

$$\begin{aligned} &|a_0^p(u_k, \varphi_k) - a_0^p(u_m, \varphi_m)| \\ &= \left| \int_{\Omega} \langle AXu_k, Xu_k \rangle^{\frac{p-2}{2}} \langle AXu_k, X\varphi_k \rangle \right. \\ &\quad \left. - \langle AXu_m, Xu_m \rangle^{\frac{p-2}{2}} \langle AXu_m, X\varphi_m \rangle dx \right| \\ &\leq \int_{\Omega} \langle AXu_k, Xu_k \rangle^{\frac{p-2}{2}} |\langle AXu_k, X(\varphi_k - \varphi_m) \rangle| dx \\ &\quad + \int_{\Omega} \left| \langle AXu_k, Xu_k \rangle^{\frac{p-2}{2}} AXu_k - \langle AXu_m, Xu_m \rangle^{\frac{p-2}{2}} AXu_m, X\varphi_m \rangle \right| dx \end{aligned}$$

$$= I_1 + I_2.$$

The first term is easy to estimate: the same computation as in (3.7) shows that

$$I_1 \leq \|Xu_k\|_A^{p-1} \|X\varphi_k - X\varphi_m\|_A,$$

and so $I \rightarrow 0$ as $k, m \rightarrow \infty$.

The estimate for I_2 depends on the size of p . If $p \geq 2$, we have the vector inequality

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq (p-1) (|\xi|^{p-2} + |\eta|^{p-2}) |\xi - \eta|,$$

where $\xi, \eta \in \mathbb{R}^m$. (This is implicit in [26, Chapter 10, p. 73].) Then, since $|A^{\frac{1}{2}}Xu| = \langle AXu, Xu \rangle^{\frac{1}{2}}$, we have that for any $u \in \text{Lip}(\overline{\Omega})$,

$$\begin{aligned} I_2 &= \int_{\Omega} \left| \langle \langle AXu_k, Xu_k \rangle^{\frac{p-2}{2}} AXu_k - \langle \langle AXu_m, Xu_m \rangle^{\frac{p-2}{2}} AXu_m, X\varphi_m \rangle \right| dx \\ &= \int_{\Omega} \left| \langle |A^{\frac{1}{2}}Xu_k|^{p-2} A^{\frac{1}{2}}Xu_k - |A^{\frac{1}{2}}Xu_m|^{p-2} A^{\frac{1}{2}}Xu_m, A^{\frac{1}{2}}X\varphi_m \rangle \right| dx \\ &\leq \int_{\Omega} \left| |A^{\frac{1}{2}}Xu_k|^{p-2} A^{\frac{1}{2}}Xu_k - |A^{\frac{1}{2}}Xu_m|^{p-2} A^{\frac{1}{2}}Xu_m \right| \left| A^{\frac{1}{2}}X\varphi_m \right| dx \\ &\leq C \int_{\Omega} \left(|A^{\frac{1}{2}}Xu_k|^{p-2} + |A^{\frac{1}{2}}Xu_m|^{p-2} \right) \left| A^{\frac{1}{2}}Xu_k - A^{\frac{1}{2}}Xu_m \right| \left| A^{\frac{1}{2}}X\varphi_m \right| dx. \end{aligned}$$

Thus, by Hölder's inequality,

$$\begin{aligned} I_2 &\leq C \left(\int_{\Omega} (|A^{\frac{1}{2}}Xu_k|^{p-2} + |A^{\frac{1}{2}}Xu_m|^{p-2})^{\frac{p}{p-2}} dx \right)^{\frac{p-2}{p}} \left(\int_{\Omega} |A^{\frac{1}{2}}X(u_k - u_m)|^p dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\Omega} |A^{\frac{1}{2}}X\varphi_m|^p dx \right)^{\frac{1}{p}} \\ &\leq C (\|Xu_k\|_A^{p-2} + \|Xu_m\|_A^{p-2}) \|Xu_k - Xu_m\|_A \|X\varphi_m\|_A, \end{aligned}$$

from which it follows that $I_2 \rightarrow 0$ as $k, m \rightarrow \infty$.

In order to estimate I_2 when $1 < p < 2$, we have a similar vector inequality,

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq c_p |\xi - \eta|^{p-1},$$

where $\xi, \eta \in \mathbb{R}^m$. (See [26, p. 43].) We can now proceed as in the case $p \geq 2$ to get

$$\begin{aligned} I_2 &\leq C \int_{\Omega} \left| |A^{\frac{1}{2}}Xu_k - A^{\frac{1}{2}}Xu_m|^{p-1} \right| \left| A^{\frac{1}{2}}X\varphi_m \right| dx \\ &\leq \|Xu_k - Xu_m\|_A^{p-1} \|X\varphi_m\|_A. \end{aligned}$$

Finally, essentially the same argument shows that if $\{\tilde{u}_k\}$ and $\{\tilde{\varphi}_k\}$ are also sequences converging to u and φ , then

$$\lim_{k \rightarrow \infty} a_0^p(u_k, \varphi_k) = \lim_{k \rightarrow \infty} a_0^p(\tilde{u}_k, \tilde{\varphi}_k).$$

This completes the proof. \square

We can now define a weak solution to the equation $\mathcal{L}_p u = 0$.

Definition 3.9. We say $u \in S^p(\Omega)$ is a weak solution of $\mathcal{L}_p u = 0$ if for all $\varphi \in S_0^p(\Omega)$, $a_0^p(u, \varphi) = 0$.

To state the Harnack inequality below we need a notion of a non-negative solution. We will say that $u \in S^p(\Omega)$ (or $S_0^p(\Omega)$) is non-negative, and write $u \geq 0$, if there exists a Cauchy sequence $\{u_j\}$ in $\text{Lip}(\overline{\Omega})$ converging to u such that $u_j(x) \geq 0$ for all x and j . Given this, we say that $u \in S^p(\Omega)$ is a weak subsolution of $\mathcal{L}_p u \leq 0$ if $a_0^p(u, \varphi) \leq 0$ for $\varphi \in S_0^p(\Omega)$, $\varphi \geq 0$.

Clearly, our definition of non-negative implies that $u(x) \geq 0$ almost everywhere. Given $u \in S^p(\Omega)$ such that $u(x) \geq 0$ a.e., we do not know *a priori* that $u \geq 0$ in the sense we have just defined, and in [6, 11] this was taken as a hypothesis in order to prove regularity results. However, if we impose a stronger condition on the weight v , then we do get that our definition coincides with the usual definition of non-negative.

Proposition 3.10. *Given a ball B and $v \in A_\infty(B, \rho, dx)$ suppose that $\phi \in S^p(B)$ is such that $\phi \geq 0$ a.e. in B . Then $\phi \geq 0$: that is, there exists a representative Cauchy sequence $\{\psi_k\}$ of ϕ such that $\psi_k \geq 0$ a.e. for all k .*

Proof. Let $\{\phi_k\} \subset \text{Lip}(\overline{\Omega})$ be a representative Cauchy sequence of ϕ . Define the sequence $\{\psi_k\}$ by

$$\psi_k(x) = \max(\phi_k(x), -1/k) + 1/k.$$

Then for all k , $\psi_k \geq 0$ and $\psi_k \in \text{Lip}(\overline{\Omega})$. It will suffice to prove that there exists a subsequence $\{\psi_{k_j}\}$ such that

$$\lim_{j \rightarrow \infty} \|\phi_{k_j} - \psi_{k_j}\| = 0;$$

it follows from this that $\{\psi_{k_j}\}$ is a representative Cauchy sequence for ϕ . Since we will have to pass to a subsequence several times, we will abuse notation and denote each successive subsequence by $\{\psi_k\}$.

We first construct a subsequence such that $\|\psi_k - \phi_k\|_{L^p(v, B)} \rightarrow 0$. Let $E_k = \{x \in B : \phi_k(x) < -1/k\}$. Then

$$\int_B |\psi_k(x) - \phi_k(x)|^p v(x) dx = k^{-p} v(B \setminus E_k) + \int_{E_k} |\phi_k(x)|^p v(x) dx.$$

The first term clearly tends to zero as $k \rightarrow \infty$. Since $\phi_k \rightarrow \phi$ in $L^p(v, B)$ norm, by the converse of the dominated convergence theorem (see [25]), there exists $\Phi \in L^1(v, B)$ such that, after we pass to a subsequence, $\phi_k \rightarrow \phi$ pointwise a.e. and $|\phi_k|^p \leq \Phi$. Therefore, by the dominated convergence theorem, to show that the second term above tends to 0, it will suffice to show that $|E_k| \rightarrow 0$ as $k \rightarrow \infty$.

Since $\phi \geq 0$, we have that

$$E_k = \{x \in B : \phi_k(x) < -1/k\} \subset \{x \in B : |\phi_k(x) - \phi(x)| > 1/k\} = F_k.$$

Since $\phi_k \rightarrow \phi$ in $L^p(v, B)$, it converges in measure with respect to the measure $v dx$, so by again passing to a subsequence we may assume that $v(F_k) < 1/k$.

By assumption $v \in A_\infty(B, \rho, dx)$, and so by Lemma 2.4,

$$\frac{|F_k|}{|B|} \leq C \left(\frac{v(F_k)}{v(B)} \right)^{1/p}.$$

Hence, $|F_k| \rightarrow 0$ as $k \rightarrow \infty$, and so $|E_k| \rightarrow 0$ as well.

We will now show that there exists a subsequence such that

$$\lim_{k \rightarrow \infty} \|X\psi_k - X\phi_k\|_A^p = \lim_{k \rightarrow \infty} \int_B \langle AX(\psi_k - \phi_k), X(\psi_k - \phi_k) \rangle^{p/2} dx = 0.$$

If $f = c$ is constant, then $Xf = 0$ a.e. Hence, on $B \setminus E_k$, $X\phi_k = X\psi_k$, and so by dividing the domain as we did above, we have to show that

$$(3.8) \quad \lim_{k \rightarrow \infty} \int_{E_k} \langle AX\phi_k, X\phi_k \rangle^{p/2} dx = 0.$$

We will actually construct a sequence of nested sets G_k such that $E_k \subset G_k$ and show that

$$(3.9) \quad \lim_{k \rightarrow \infty} \int_{G_k} \langle AX\phi_k, X\phi_k \rangle^{p/2} dx = 0.$$

Since $\langle AX\phi_k, X\phi_k \rangle = |A^{1/2}X\phi_k|^2$, the integrand is positive and so (3.8) follows from (3.9).

Since $|E_k| \rightarrow 0$, by passing to a subsequence, we may assume that $|E_{k+1}| \leq \frac{1}{2}|E_k|$. Define

$$G_k = \bigcup_{j \geq k} E_j.$$

Then $|G_k| \leq 2|E_k|$, and so $|G_k| \rightarrow 0$ as $k \rightarrow \infty$.

Since ϕ_k is Cauchy with respect to the $S^p(B)$ norm, we know that the sequence of functions $f_k = \langle AX\phi_k, X\phi_k \rangle^{p/2}$ is Cauchy in $L^1(B)$ and therefore converges to some function $f \in L^1(B)$. Fix $\epsilon > 0$. Then there exists k_0 such that

$$\|f\chi_{G_{k_0}}\|_{L^1(B)} < \epsilon/2.$$

But for fixed k_0 , we have that $f_k\chi_{G_{k_0}} \rightarrow f\chi_{G_{k_0}}$ in $L^1(B)$. Hence, for all $k \geq k_0$ sufficiently large, we have that

$$\|f_k\chi_{G_k}\|_{L^1(B)} \leq \|f_k\chi_{G_{k_0}}\|_{L^1(B)} < \epsilon.$$

Since ϵ is arbitrary, we get (3.9) and our proof is complete. \square

Finally, we show that weak solutions to $\mathcal{L}_p u = 0$ exist. When $p = 2$ this is a straightforward application of the Lax-Milgram theorem. For all $p > 1$ we need a more delicate argument.

Theorem 3.11. *Suppose (w, v) is a pair of weights satisfying condition (1) of Definition 3.1 such that the global Sobolev inequality (3.3) holds. Let $A = A(x)$ be an $m \times m$ real symmetric matrix defined in Ω that satisfies the degenerate ellipticity condition (3.5). Then for all $\psi \in S^p(\Omega)$, there exists $u \in S^p(\Omega)$ such that u is a weak solution of $\mathcal{L}_p u = 0$ in Ω and $u - \psi \in S_0^p(\Omega)$.*

For the proof of Theorem 3.11 we will use an abstract result taken from Kinderlehrer and Stampacchia [23]. Let V be a reflexive Banach space with norm $\|\cdot\|$ and dual V' , and denote by (\cdot, \cdot) a pairing between V' and V . Given a nonempty closed convex subset $U \subset V$ and a mapping $\mathcal{A} : U \rightarrow V'$, \mathcal{A} is monotone if

$$(\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2) \geq 0, \quad u_1, u_2 \in U,$$

and \mathcal{A} is coercive if there exists $g \in U$ such that

$$\frac{(\mathcal{A}u_j - \mathcal{A}g, u_j - g)}{\|u_j - g\|} \rightarrow \infty$$

for all sequences $\{u_j\} \subset U$ such that $\|u_j\| \rightarrow \infty$.

Proposition 3.12. *Let U be a nonempty closed convex subset of a reflexive Banach space V and let $\mathcal{A} : U \rightarrow V'$ be monotone, coercive, and weakly continuous on U . Then there exists $u \in U$ such that*

$$\langle \mathcal{A}u, g - u \rangle \geq 0, \quad g \in U.$$

Proof of Theorem 3.11. With the notation of Proposition 3.12 we set $V = S^p(\Omega)$, $U = \{u \in S^p(\Omega) : u - \psi \in S_0^p(\Omega)\}$, and define $\mathcal{A}u(\cdot) = a_0^p(u, \cdot)$ for $u \in U$. Recall that $S^p(\Omega)$ is a reflexive Banach space. Furthermore, it is clear that U is a nonempty (since $\psi \in U$), convex closed subset of $S^p(\Omega)$. We claim that $\mathcal{A}u \in S^p(\Omega)'$ for all $u \in U$ (in fact for all $u \in S^p(\Omega)$). First, $a_0^p(u, \cdot)$ is clearly linear. Further, by using (3.7) for functions in $\text{Lip}(\overline{\Omega})$ and then passing to the limit using Lemma 3.8, we have that for all $u, g \in S^p(\Omega)$,

$$|a_0^p(u, g)| \leq \|u\|^{p-1} \|g\|.$$

To prove that \mathcal{A} is monotone, let $u_1, u_2 \in \text{Lip}(\overline{\Omega})$; then a straightforward computations gives

$$\begin{aligned} (\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2) &= a_0^p(u_1, u_1 - u_2) - a_0^p(u_2, u_1 - u_2) \\ &= \int_{\Omega} \langle |A^{\frac{1}{2}} X u_1|^{p-2} A^{\frac{1}{2}} X u_1 - |A^{\frac{1}{2}} X u_2|^{p-2} A^{\frac{1}{2}} X u_2, A^{\frac{1}{2}} X u_1 - A^{\frac{1}{2}} X u_2 \rangle dx. \end{aligned}$$

Since

$$\langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq 0, \quad \xi, \eta \in \mathbb{R}^m,$$

it follows that

$$(3.10) \quad (\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2) \geq 0.$$

By Lemma 3.8 we conclude that (3.10) holds for all $u_1, u_2 \in S^p(\Omega)$.

To show that \mathcal{A} is weakly continuous again fix $u_1, u_2, g \in \text{Lip}(\overline{\Omega})$. Then

$$\begin{aligned} |(\mathcal{A}u_1, g) - (\mathcal{A}u_2, g)| &= a_0^p(u_1, g) - a_0^p(u_2, g) \\ &= \int_{\Omega} \langle |A^{\frac{1}{2}}Xu_1|^{p-2}A^{\frac{1}{2}}Xu_1 - |A^{\frac{1}{2}}Xu_2|^{p-2}A^{\frac{1}{2}}Xu_2, A^{\frac{1}{2}}Xg \rangle dx. \end{aligned}$$

If we repeat the argument used to estimate the term I_2 in Lemma 3.8, we get

$$|(\mathcal{A}u_1, g) - (\mathcal{A}u_2, g)| \leq C(\|u_1\|^{p-2} + \|u_2\|^{p-2}) \|u_1 - u_2\| \|g\|, \quad p \geq 2,$$

and

$$|(\mathcal{A}u_1, g) - (\mathcal{A}u_2, g)| \leq C\|u_1 - u_2\|^{p-1}\|g\|, \quad 1 < p < 2.$$

Therefore, passing to the limit we have that these inequalities hold for any $u_1, u_2, g \in S^p(\Omega)$. Hence, \mathcal{A} is weakly continuous on $S^p(\Omega)$: if $\{u_j\} \subset S^p(\Omega)$ is such that $u_j \rightarrow u$ then $(\mathcal{A}u_j, g) \rightarrow (\mathcal{A}u, g)$ for all $g \in S^p(\Omega)$.

Finally, we show that \mathcal{A} is coercive. We will prove that

$$(3.11) \quad \frac{(\mathcal{A}u_j - \mathcal{A}\psi, u_j - \psi)}{\|u_j - \psi\|} \rightarrow \infty$$

for all sequences $\{u_j\} \subset U$ such that $\|u_j\| \rightarrow \infty$, where ψ is as in the statement of the theorem. Since the global Sobolev inequality holds we have that $\|u_j - \psi\| \approx \|u_j - \psi\|_A$, so we can use the $S_0^p(\Omega)$ norm in the denominator. Further, by a limiting argument using Lemma 3.8 it suffice to prove this for $u, \psi \in \text{Lip}(\overline{\Omega})$.

We first consider the case $p \geq 2$. Arguing as we did before, we have that

$$\begin{aligned} (\mathcal{A}u - \mathcal{A}\psi, u - \psi) &= a_0^p(u, u - \psi) - a_0^p(\psi, u - \psi) \\ &= \int_{\Omega} \left| \langle |A^{\frac{1}{2}}Xu|^{p-2}A^{\frac{1}{2}}Xu - |A^{\frac{1}{2}}X\psi|^{p-2}A^{\frac{1}{2}}X\psi, A^{\frac{1}{2}}Xu - A^{\frac{1}{2}}X\psi \rangle \right| dx. \end{aligned}$$

Since $p \geq 2$, we have that

$$(3.12) \quad \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \geq 2^{2-p}|\xi - \eta|^p, \quad \xi, \eta \in \mathbb{R}^m.$$

(See [26, Chapter 10] for a proof of (3.12)). Hence,

$$\begin{aligned} (\mathcal{A}u - \mathcal{A}\psi, u - \psi) &\geq 2^{2-p} \int_{\Omega} \left| A^{\frac{1}{2}}Xu - A^{\frac{1}{2}}X\psi \right|^p dx \\ &= 2^{2-p} \|u - \psi\|_A^p, \end{aligned}$$

and (3.11) follows immediately.

When $1 < p \leq 2$ we argue as before, except that we use the inequality

$$(3.13) \quad \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \geq |\xi - \eta|^p - |\eta|^{p-1}|\xi - \eta|, \quad \xi, \eta \in \mathbb{R}^m.$$

(Inequality (3.13) follows in a similar manner to (3.12).) Combining this with Hölder's inequality, we have that

$$(\mathcal{A}u - \mathcal{A}\psi, u - \psi)$$

$$\begin{aligned}
&\geq \int_{\Omega} \left| A^{\frac{1}{2}} Xu - A^{\frac{1}{2}} X\psi \right|^p dx - \int_{\Omega} \left| A^{\frac{1}{2}} X\psi \right|^{p-1} \left| A^{\frac{1}{2}} Xu - A^{\frac{1}{2}} X\psi \right| dx \\
&\geq \|u - \psi\|_A^p - \|\psi\|_A^{p-1} \|u - \psi\|_A.
\end{aligned}$$

Again, (3.11) follows immediately.

Therefore, by Proposition 3.12 there exists $u \in U$ such that

$$a_0^p(u, g - u) \geq 0$$

for all $g \in U$. If $\varphi \in S_0^p(\Omega)$, then $u + \varphi$ and $u - \varphi$ belong to U and therefore

$$a_0^p(u, \varphi) \geq 0 \quad \text{and} \quad -a_0^p(u, \varphi) \geq 0,$$

from which it follows that $a_0^p(u, \varphi) = 0$. Therefore, we have shown that u is a weak solution of $\mathcal{L}_p u = 0$ such that $u - \psi \in S_0^p(\Omega)$. \square

3.4. Harnack inequality. Central to the proofs of our main results are a mean value inequality and a Harnack inequality for weak solutions of the equation $\mathcal{L}_p u = 0$. As always, $A = A(x)$ is a real, $m \times m$ symmetric matrix defined in Ω that satisfies the degenerate ellipticity condition (3.5).

Lemma 3.13. *Given p , $1 < p < \infty$, a pair of p -admissible weights (w, v) in Ω , and a compact set $K \subset \Omega$, let $r_0 > 0$ be as in Theorem 3.3. Then given any ball B with center in K and $r(B) < r_0$, suppose $u \in S^p(B)$ is a weak subsolution of $\mathcal{L}_p u \leq 0$. Then there exist constants c and d , depending only on the weights, such that, for any α , $1/2 \leq \alpha < 1$,*

$$\left(\operatorname{ess\,sup}_{x \in \alpha B} u^+(x) \right)^p \leq \frac{c}{(1 - \alpha)^d} \mu_p^{\frac{p\sigma}{\sigma-1}} \frac{1}{v(B)} \int_B u^+(x)^p v(x) dx,$$

where $\sigma = q/p > 1$ with q as in condition (2) of Definition 3.1, $u^+ = \max(u, 0)$, and

$$\mu_p = \mu_p(B) = \left(\frac{v(B)}{w(B)} \right)^{1/p}.$$

Remark 3.14. We may restate Lemma 3.13 as follows: given any $x \in \Omega$, then for all balls centered at x whose radius is sufficiently small, the mean value inequality holds.

Lemma 3.13 was proved in [11] for non-negative weak sub-solutions, with u^+ replaced by u . However, the proof can be readily adapted: for details on the minor changes required, see [6] where the argument is done for the case $p = 2$; the same changes apply for arbitrary $p > 1$. As a consequence, we get the following result.

Theorem 3.15. *With the same hypotheses as Lemma 3.13, suppose that $u \in S^p(B)$ is a weak solution of $\mathcal{L}_p u = 0$. Then*

$$\operatorname{ess\,sup}_{x \in \alpha B} |u(x)|^p \leq \frac{c}{(1 - \alpha)^d} \mu_p^{\frac{p\sigma}{\sigma-1}} \frac{1}{v(B)} \int_B |u(x)|^p v(x) dx.$$

Proof. Since u is a weak solution, then u and $-u$ are both weak subsolutions, so the conclusion of Lemma 3.13 holds for both of these functions. Since

$$|u| = \max(u^+, (-u)^+),$$

the desired inequality follows at once. \square

Our final result is a Harnack inequality, which is also proved in [11].

Theorem 3.16. *Given p , $1 < p < \infty$, a pair of p -admissible weights (w, v) in Ω , and a compact set $K \subset \Omega$, let $r_0 > 0$ be as in Theorem 3.3. Given any ball B with center in K and $r(B) < r_0/2$, suppose u is a non-negative weak solution of $\mathcal{L}_p u = 0$ in $S^p(2B)$. Then*

$$\operatorname{ess\,sup}_B u \leq e^{C_K \mu_p(B)} \operatorname{ess\,inf}_B u$$

with $\mu_p(B) = (v(B)/w(B))^{1/p}$ and C_K independent of u and B .

Remark 3.17. As with the previous two results, we may rephrase Theorem 3.16 as saying that given $x \in B$, the Harnack inequality holds for all balls B centered at x with radius sufficiently small. This is precisely what we need to perform Moser iteration below.

Remark 3.18. In [11], Theorem 3.16 is proved for solutions u that are non-negative in the sense that there exists a sequence $\{u_j\}$ of non-negative Lipschitz functions converging to u . By Proposition 3.10, when we apply this result below it will suffice to assume that u is a non-negative in the sense that $u(x) \geq 0$ a.e.

4. PROOF OF MAIN RESULTS

In this section we state and prove our main results. Throughout, let Ω be an open, bounded and connected subset of \mathbb{R}^n or let $\Omega = \mathbb{R}^n$. Let $X = (X_1, \dots, X_m)$ be a C^∞ vector field that satisfies Hörmander's ellipticity condition in \mathbb{R}^n , ρ be the corresponding Carnot-Carathéodory metric in \mathbb{R}^n , and Q be the homogeneous dimension of \mathbb{R}^n with respect to X .

Our first result is a generalization of Theorem A in the Introduction.

Theorem 4.1. *Given Ω , X , and p , $1 < p < \infty$, let A be an $m \times m$ matrix of measurable functions defined in Ω that satisfies the ellipticity condition*

$$(4.1) \quad \lambda w(x)^{2/p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda w(x)^{2/p} |\xi|^2, \quad \xi \in \mathbb{R}^m, x \in \Omega,$$

where $w \in A_p(\Omega, \rho, dx)$ and $0 < \lambda < \Lambda < \infty$. Then every weak solution u of the equation

$$(4.2) \quad \mathcal{L}_p u = \operatorname{div}_X(\langle AXu, Xu \rangle^{\frac{p-2}{2}} AXu) = 0$$

is Hölder continuous on compact subsets of Ω .

Remark 4.2. Recall that by Remark 2.2 if Ω satisfies certain boundary conditions, then we can assume $w \in A_p(\mathbb{R}^n, \rho, dx)$. Conversely, by Wolff's extension theorem, in certain cases, $w \in A_p(\Omega, \rho, dx)$ can be extended to an A_p weight on \mathbb{R}^n . See [14] for more details.

Our next two results generalize Theorem B from the Introduction. We have two results depending on whether we are working in a general Carnot-Carathéodory space or if we are in the special case of a Carnot group. To state both results, we need a definition.

Definition 4.3. Given a function $u \in L^1_{loc}$, we say that u has an approximate limit at a point x if there exists $z \in \mathbb{R}$ such that

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |u(y) - z| dy = 0.$$

Note that by the Lebesgue differentiation theorem, this limit exists almost everywhere and we have $z = u(x)$. At each point x where u has an approximate limit, define the function $\tilde{u}(x) = z$. The function \tilde{u} is called the precise representative of u .

Theorem 4.4. *Given Ω , X and p , $1 < p < \infty$, let A be an $m \times m$ matrix of measurable functions defined in Ω that satisfies the ellipticity condition*

$$(4.3) \quad k(x)^{-2/p'} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq k(x)^{2/p} |\xi|^2, \quad \xi \in \mathbb{R}^m, x \in \Omega,$$

where

$$k \in A_{p'}(\Omega, \rho, dx) \cap RH_\tau(\Omega, \rho, dx), \quad \tau = 1 + \frac{p(Q-1)}{n+p-Q}.$$

Then every weak solution of (4.2) is continuous almost everywhere. More precisely, \tilde{u} , the precise representative of u , is continuous at every point of the set

$$S(k) = \{x \in \Omega : Mk(x) < \infty\}.$$

When X generates a homogeneous Carnot group, we have the following theorem.

Theorem 4.5. *Given $\Omega = \mathbb{R}^n$, X and p , $1 < p < \infty$, suppose X generates a homogeneous Carnot group. Let A be an $m \times m$ matrix of measurable functions that satisfies the ellipticity condition (4.3), where $k \in A_{p'}(\Omega, \rho, dx) \cap RH_Q(\Omega, \rho, dx)$. Then every weak solution of (4.2) is continuous almost everywhere: the precise representative \tilde{u} is continuous at every point of the set*

$$S(k) = \{x \in \mathbb{R}^n : Mk(x) < \infty\}.$$

Remark 4.6. Note that since $Q \geq n$, $\tau \geq Q$. Therefore, the hypotheses on the weight k in Theorem 4.5 is weaker than in Theorem 4.4.

Remark 4.7. Also notice that since $k \in RH_n$, Mk is an A_1 weight. This shows that the set $S(k)$ has measure zero, and hence the weak solutions to $\mathcal{L}_p u = 0$ are continuous almost everywhere.

Proof of Theorem 4.1. We want to use the Harnack inequality, Theorem 3.16, with $v = w$ and apply Moser iteration. To do so, we first have to prove that $(\lambda^{p/2}w, \Lambda^{p/2}w)$ is a pair of p -admissible weights. Since $w \in A_p(\Omega, \rho, dx)$, so is $\lambda^{p/2}w$, and by Remark 2.6, $\Lambda^{p/2}w$ is locally doubling. We also have that condition (2) holds. Indeed, let K be a compact subset

of Ω , R_K as given by Lemma 2.1, B a metric ball with center in K and $r(B) \leq R_K$, and $B(x_1, r_1) \subset B(x_2, r_2) \subset B$. By Lemma 2.1 it is enough to show that there exists $q > p$ such that

$$(4.4) \quad \left(\frac{|B(x_1, r_1)|}{|B(x_2, r_2)|} \right)^{1/Q} \leq C \left(\frac{w(B(x_1, r_1))}{w(B(x_2, r_2))} \right)^{\frac{1}{p} - \frac{1}{q}}.$$

By Lemma 2.3 and Remark 2.5 there exists $\epsilon > 0$, independent of B , such that $w \in A_{p-\epsilon}(B, \rho, dx)$ with uniform constants. Since $Q \geq n \geq 2$, we may assume that ϵ is so small that $Q > p/(p - \epsilon)$. Therefore, we can define $q > p$ by

$$\frac{1}{p} - \frac{1}{Q(p - \epsilon)} = \frac{1}{q},$$

and by Lemma 2.4 (with $p - \epsilon$ in place of p and $E = B(x_1, r_1)$) and Remark 2.5 we get (4.4) with uniform constants. Hence, $(\lambda^{p/2}w, \Lambda^{p/2}w)$ is a pair of p -admissible weights.

Any solution now satisfy the hypotheses of Theorems 3.15 and 3.16. In particular, solutions are locally bounded and non-negative solutions satisfy a locally uniform Harnack inequality. It is well known that Hölder continuity easily follows from these assumptions, see [15, Chap. 8]. □

The proofs of Theorems 4.4 and 4.5 follow the same basic outline as the proof of Theorem 4.1. A key step in both, however, is proving that $(k^{-p/p'}, k)$ is a pair of p -admissible weights. By assumption, $k \in A_{p'}(\Omega, \rho, dx)$, so $k^{-p/p'} = k^{1-p} \in A_p(\Omega, \rho, dx)$. That condition (2) holds is the substance of the next two results.

Theorem 4.8. *The pair $(k^{-p/p'}, k)$ satisfies condition (2) of Definition 3.1 provided that $k^{-p/p'} \in A_s(\Omega, \rho, dx)$, $k \in RH_t(\Omega, \rho, dx)$, $1 < t < \infty$, $\frac{p}{Q} < s < \infty$, and*

$$(4.5) \quad \frac{pt'}{n} \leq \left(\frac{sQ}{p} \right)' - 1.$$

In particular, this is the case if

$$(4.6) \quad k \in A_{p'}(\Omega, \rho, dx) \cap RH_\tau(\Omega, \rho, dx), \quad \tau = 1 + \frac{p(Q-1)}{n+p-Q}.$$

If X generates a Carnot group on \mathbb{R}^n , then we can repeat the proof of Theorem 4.8 with n replaced by Q to get the following result.

Theorem 4.9. *Suppose $\Omega = \mathbb{R}^n$ and X generates a homogeneous Carnot group. Then the pair $(k^{-p/p'}, k)$ satisfies condition (2) of Definition 3.1 with uniform constants for all compact sets, provided that $k^{-p/p'} \in A_s(\mathbb{R}^n, \rho, dx)$, $k \in RH_t(\mathbb{R}^n, \rho, dx)$, $1 < t < \infty$,*

$\frac{p}{Q} < s < \infty$, and

$$\frac{pt'}{Q} \leq \left(\frac{sQ}{p} \right)' - 1.$$

In particular, this is the case if

$$k \in A_{p'}(\mathbb{R}^n, \rho, dx) \cap RH_Q(\mathbb{R}^n, \rho, dx),$$

which in turn implies that $k^{-p/p'} \in A_{1+\frac{p-1}{Q}}(\mathbb{R}^n, \rho, dx)$.

Proof of Theorem 4.8. Let K be a compact subset of Ω ; R_K be given by Lemma 2.1; B be a metric ball with center in K and radius $r(B) \leq R_K$; and $B_1 = B(x_1, r_1)$, $B_2 = B(x_2, r_2)$ be such that $B_1 \subset B_2 \subset B$. Fix a pair $(k^{-p/p'}, k)$ as in the hypotheses. Then by (4.5),

$$(4.7) \quad \frac{n}{pt'} \geq \frac{sQ}{p} - 1.$$

By Lemma 2.3, since the restriction of $k^{-p/p'}$ to B belongs to $A_s(B, \rho, dx)$, there exists $\epsilon > 0$ such that the restriction of $k^{-p/p'}$ to B is in $A_{s-\epsilon}(B, \rho, dx)$. Furthermore, by Remark 2.5, we may choose ϵ independent of B and the constants are uniform. Combining this with (4.7) we get that

$$\frac{n}{p} \left(1 - \frac{1}{t} \right) > \frac{(s-\epsilon)Q}{p} - 1.$$

Rearranging terms we define

$$(4.8) \quad q = \frac{n(t')^{-1}}{(s-\epsilon)Qp^{-1} - 1} > p.$$

Therefore, since $k \in RH_t(B, \rho, dx)$, by Lemmas 2.1 and 2.4 and inequality (4.8),

$$\begin{aligned} \frac{r_1}{r_2} \left(\frac{k(B_1)}{k(B_2)} \right)^{1/q} &\leq C \frac{r_1}{r_2} \left(\frac{|B_1|}{|B_2|} \right)^{\frac{1}{qt'}} \leq C \left(\frac{r_1}{r_2} \right)^{\frac{n}{qt'} + 1} \\ &= C \left(\frac{r_1}{r_2} \right)^{\frac{(s-\epsilon)Q}{p}} \leq C \left(\frac{|B_1|}{|B_2|} \right)^{\frac{s-\epsilon}{p}} \leq C \left(\frac{k^{-p/p'}(B_1)}{k^{-p/p'}(B_2)} \right)^{1/p}, \end{aligned}$$

again with uniform constants by Remark 2.5. Since this is true for every such ball B , $(k^{-p/p'}, k)$ satisfies (2) of Definition 3.1.

We now prove that (4.6) implies (4.5). By Lemma 2.7,

$$k^\tau \in A_{\tau(p'-1)+1}(B, \rho, dx).$$

Let $\sigma = \tau(p' - 1) + 1$. Then a straightforward calculation shows that

$$\sigma' = \frac{\tau(p' - 1) + 1}{\tau(p' - 1)} = \frac{p^2 + np - p}{n - Q + pQ},$$

and by Lemma 2.3,

$$k^{-p/p'} = k^{\tau(1-\sigma')} \in A_{\sigma'}(B, \rho, dx).$$

On the other hand if we replace t by τ in (4.5) and take equality, then solving for s we get (after another calculation)

$$s = \frac{n}{Q} \left(1 - \frac{1}{\tau}\right) + \frac{p}{Q} = \frac{p^2 + np - p}{n - Q + pQ} = \sigma'.$$

Note that $s > \frac{p}{Q}$ since $Q \geq n > 1$. Therefore, $(k^{-p/p'}, k)$ satisfies condition (2) of Definition 3.1 and our proof is complete. \square

Proof of Theorems 4.4 and 4.5. Our proof is a generalization of the proof of the main theorem in [7]. By Theorems 4.8, 4.9, and part (a) of Lemma 2.3 the pair of weights $(k^{-p/p'}, k)$ are p -admissible in Ω . Let $B(x, r_0) \subseteq \Omega$ be a ball with sufficiently small radius so that Theorems 3.15 and 3.16 hold. Let $B = B(x, r)$ where $r \leq r_0$ and set

$$M = \operatorname{ess\,sup}_{\frac{1}{2}B} u, \quad m = \operatorname{ess\,inf}_{\frac{1}{2}B} u, \quad M' = \operatorname{ess\,sup}_B u, \quad m = \operatorname{ess\,inf}_B u.$$

Arguing as in the proof of Theorem 4.1 we obtain

$$\operatorname{osc}_u(x, r/2) \leq \frac{\exp(C_{r_0}(x)\mu_p(\frac{1}{2}B)) - 1}{\exp(C_{r_0}(x)\mu_p(\frac{1}{2}B)) + 1} \operatorname{osc}_u(x, r),$$

where the constant $C_{r_0}(x)$ depends on x and r_0 , but is finite for each x . For any ball B containing x , Hölder's inequality yields

$$\mu_p(B) = \left(\frac{\int_B k \, dy}{\int_B k^{-p/p'} \, dy} \right)^{1/p} \leq \int_B k \, dx \leq Mk(x);$$

hence,

$$(4.9) \quad \operatorname{osc}_u(x, r/2) \leq \frac{\exp(C_{r_0}(x)Mk(x)) - 1}{\exp(C_{r_0}(x)Mk(x)) + 1} \operatorname{osc}_u(x, r) = \gamma(x) \operatorname{osc}_u(x, r).$$

We will now prove that the precise representative of u is continuous on the set $S(k)$. Notice that

$$S(k) = \{x \in \Omega : Mk(x) < \infty\} = \{x \in \Omega : \gamma(x) < 1\}$$

and fix $x \in S(k)$. Therefore, since $\gamma(x) < 1$, we may apply Lemma 8.23 in [15] to inequality (4.9) to obtain

$$\operatorname{osc}_u(x, r) \leq c(x) \left(\frac{r}{r_0} \right)^{\alpha(x)}, \quad 0 < r \leq r_0,$$

where $c(x)$ and $\alpha(x)$ are both finite constants depending on x . Set

$$u_{B(x,r)} = \int_{B(x,r)} u(y) \, dy$$

and notice that $\lim_{r \rightarrow 0} u_{B(x,r)}$ exists. Indeed if $0 < s < t < r_0$, then

$$\begin{aligned} |u_{B(x,t)} - u_{B(x,s)}| &\leq \int_{B(x,t)} \int_{B(x,s)} |u(z) - u(y)| dy dz \\ &\leq \text{osc}_u(x, t) \\ &\leq c(x) \left(\frac{t}{r_0} \right)^{\alpha(x)} \rightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned}$$

If we set $\tilde{u}(x) = \lim_{r \rightarrow 0} u_{B(x,r)}$ then a similar argument shows that \tilde{u} is the precise representative of u . To see that \tilde{u} is continuous on $S(k)$, let B be a ball in Ω with sufficiently small radius. Then notice that for any $y \in S(k) \cap B$ we have

$$\text{ess inf}_B u \leq \tilde{u}(y) = \lim_{r \rightarrow 0} \int_{B(y,r)} u(z) dz \leq \text{ess sup}_B u,$$

which implies that the pointwise oscillation of \tilde{u} on $S(k) \cap B$ is dominated by the essential oscillation of u on B . The continuity of \tilde{u} on $S(k)$ follows at once. \square

5. EXAMPLES

We now show that Theorem 4.4 is sharp in the sense that there exist Ω , X , A , k and p satisfying its hypothesis and a solution u to the corresponding equation that fails to be continuous in a nonempty set of measure zero. We will work in the euclidean setting, that is, $X = \nabla$ and address the case $p = 2$. In this setting the ellipticity condition (4.3) is given by

$$(5.1) \quad k(x)^{-1} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq k(x) |\xi|^2.$$

and (4.2) becomes

$$(5.2) \quad \text{div}(A \nabla u) = 0.$$

The examples presented below are based on those given in Zhong [35]. We give an example when $n \geq 3$. An example when $n = 2$ can also be constructed in a similar fashion using another example from [35].

Example 5.1. Let $\varepsilon > 0$ and $\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| < \frac{1}{e}\} = B(0, 1/e)$. Defined $\Omega_1 = \{x \in \Omega : 2|x_n| > |x|\}$, $\Omega_2 = \{x \in \Omega : 2|x_n| < |x|\}$, and $A(x) = \tau(x) Id$, where Id is the $n \times n$ identity matrix and

$$\tau(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ |\log |x||^{-(1+\varepsilon)} & \text{if } x \in \Omega_2. \end{cases}$$

Zhong [35, Theorem 1.2] proved that the equation (5.2) has a weak solution $u \in S^2(\Omega)$ that is discontinuous at the origin. In order to prove the sharpness of Theorem 4.4 it is enough to check that there exists $\varepsilon > 0$ such that the corresponding matrix A satisfies the

degenerate ellipticity condition (5.1) and some $k \in A_2(\Omega, \rho, dx) \cap RH_n(\Omega, \rho, dx)$, where ρ denotes Euclidean distance. We have that

$$\langle A(x)\xi, \xi \rangle = \begin{cases} |\xi|^2 & \text{if } x \in \Omega_1, \\ |\log |x||^{-(1+\varepsilon)} |\xi|^2 & \text{if } x \in \Omega_2, \end{cases}$$

which implies

$$\min(1, |\log |x||^{-(1+\varepsilon)}) |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \max(1, |\log |x||^{-(1+\varepsilon)}) |\xi|^2$$

for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Since $\min(1, |\log |x||^{-(1+\varepsilon)}) = |\log |x||^{-(1+\varepsilon)}$ and $\max(1, |\log |x||^{-(1+\varepsilon)}) = 1 \leq |\log |x||^{1+\varepsilon}$, $x \in \Omega$, we obtain

$$k(x)^{-1} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq k(x) |\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n,$$

with $k(x) = |\log |x||^{1+\varepsilon}$.

We now show that, for any $\varepsilon > 0$, $k \in A_2(\Omega, \rho, dx) \cap RH_n(\Omega, \rho, dx)$. Consider the function $h(x) = |\log |x||$ if $|x| < 1/e$ and 1 otherwise. Then $h^s \in A_1(\mathbb{R}^n, \rho, dx)$ for all $s \geq 1$: that is,

$$(5.3) \quad \frac{1}{|B|} \int_B h(x)^s dx \sim \inf_{x \in B} h(x)^s.$$

The proof of (5.3) is elementary by considering balls $B(x_0, r)$ with $|x_0| < cr$ and $|x_0| \geq cr$. Note that this says that $h \in RH_s(\mathbb{R}^n, \rho, dx)$ for all $s \geq 1$. Given (5.3), we use that $|B \cap \Omega| \sim |B|$ for all Euclidean balls B with center in Ω and radius $r(B) \leq C \text{diam}(\Omega)$ to get that the restriction of h^s to Ω is in $A_1(\Omega, \rho, dx)$ for all $s \geq 1$. In particular, taking $s = 1+\varepsilon$, gives that $k \in A_1(\Omega, \rho, dx)$ and taking $s = (1+\varepsilon)n$ yields that $k^n \in A_1(\Omega, \rho, dx)$, from which we conclude that $k \in A_2(\Omega, \rho, dx) \cap RH_n(\Omega, \rho, dx)$.

6. APPLICATION TO MAPPINGS OF FINITE DISTORTION

Recall that $f : \Omega \rightarrow \mathbb{R}^n$ with $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$ is a mapping of finite distortion if there is a function $K \geq 1$ finite a.e. such that

$$|Df(x)|^n \leq K(x) J_f(x)$$

where $|Df(x)| = \sup_{|h|=1} |Df(x)h|$ is the operator norm and $J_f(x) = \det Df(x)$ is locally integrable. Throughout this section we will assume that $n \geq 3$, as much of the planar theory is known.

The study of continuity of such functions is an active line of research initiated by Vodop'janov and Gol'dšteĭn [34]. They showed that if $f \in W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$, then f has a representative that is continuous on Ω . More recent results include those of Manfredi [28], Heinonen and Koskela [18], Iwaniec, Koskela and Onninen [19] and Kauhanen, Koskela, Malý and Zhong [22]. Without assuming higher integrability on $|Df|$, many of the continuity results assume some sort of exponential integrability condition on K , namely $\exp(K) \in L_{loc}^p(\Omega)$ for some $0 < p < \infty$. Broadly speaking, to obtain continuity

one should not stray too far from the assumptions $f \in W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$ or $K \in \text{Exp } L$, (see [22]).

Our results confirm this perception. We are able to work below the natural thresholds of $W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$ and exponential integrability of the distortion function. But in doing so we do not obtain functions which are continuous everywhere, but rather continuous except on a set of measure zero. Our results are in the spirit of those in [28] where it was shown that given a mapping of finite distortion $f \in W_{loc}^{1,p}(\Omega, \mathbb{R}^n)$ for some $p, n-1 < p < n$, then f is continuous except on a set of p -capacity zero. On the other hand, our natural assumption is that $f \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$ and the distortion function belongs to an appropriate weight class.

To state our results we briefly sketch some basic facts about mappings of finite distortion. A more detailed exposition can be found in [20]. Given a mapping of finite distortion f , we will be concerned with two distortion functions: the inner, denoted K_I , and the outer, denoted K_O . They are defined as follows

$$K_I(x) = \frac{|\text{adj } Df(x)|^n}{J_f(x)^{n-1}} \quad \text{and} \quad K_O(x) = \frac{|Df(x)|^n}{J_f(x)}.$$

Here $\text{adj } A$ denotes the adjoint of the matrix A defined by the equation $A \text{adj } A = (\det A) \text{Id}$. The distortion tensor

$$G(x) = \frac{Df(x)^T Df(x)}{J_f(x)^{2/n}}$$

satisfies the ellipticity condition

$$(6.1) \quad K_O(x)^{-2/n} |\xi|^2 \leq \langle G(x)^{-1} \xi, \xi \rangle \leq K_I(x)^{2/n} |\xi|^2.$$

Since in general $K_I(x)^{1/(n-1)} \leq K_O(x) \leq K_I(x)^{n-1}$, we have the following ellipticity condition on the inner distortion function:

$$(6.2) \quad K_I(x)^{-2/n'} |\xi|^2 \leq \langle G(x)^{-1} \xi, \xi \rangle \leq K_I(x)^{2/n} |\xi|^2.$$

But this is precisely the ellipticity condition (4.3) when $p = n$ and $k = K_I$.

Theorem 6.1. *Suppose $f : \Omega \rightarrow \mathbb{R}^n$ is a mapping of finite distortion whose coordinate functions belong to $S^n(\Omega) \cap W_{loc}^{1,n-1}(\Omega)$ and whose inner distortion function satisfies $K_I \in A_{n'}(\Omega) \cap RH_n(\Omega)$. (Equivalently, $K_I^n \in A_{n'+1}(\Omega)$.) Then f has a representative whose points of continuity are the set*

$$\{x \in \Omega : MK_I(x) < \infty\}.$$

Remark 6.2. The assumption $f = (f^1, \dots, f^n) \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$ is the very minimal assumption to give meaning to the equation

$$\text{div}(\langle G(x)^{-1} \nabla f^i, \nabla f^i \rangle^{\frac{n-2}{2}} G(x)^{-1} \nabla f^i) = 0 \quad i = 1, \dots, n$$

(see [20, p. 379]). Since we are also assuming that $f^i \in S^n(\Omega)$ the assumption $f \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$ is superfluous if $K_O \in L_{loc}^{n-1}(\Omega)$. In this case $S^n(\Omega)$ embeds into $W_{loc}^{1,n-1}(\Omega)$.

Indeed, if $g \in S^n(\Omega)$ and $K_O \in L_{loc}^{n-1}(\Omega)$ then the ellipticity condition (6.1) implies

$$\left(\int_B |\nabla g|^{n-1} dx \right)^{1/(n-1)} \leq \left(\int_B |\nabla g|^n K_O^{-1} dx \right)^{1/n} \left(\int_B K_O^{n-1} dx \right)^{1/n(n-1)} \leq C \|\nabla g\|_{G^{-1}}$$

(see [18] for similar embeddings).

Proof. We will show that each of the coordinate functions $f = (f^1, \dots, f^n)$ are weak solutions to the equation $\mathcal{L}_n u = \operatorname{div}(\langle A \nabla u, \nabla u \rangle^{\frac{n-2}{2}} A \nabla u) = 0$, where $A(x) = G(x)^{-1}$. Fix a coordinate function f^i , and let u_k be a sequence of functions in $\operatorname{Lip}(\overline{\Omega})$ converging to f^i in the norm of $S^n(\Omega)$, and for the moment let $\varphi \in C_0^\infty$. By Lemma 3.8 we have

$$a_0^n(f^i, \varphi) = \lim_{k \rightarrow \infty} a_0^n(u_k, \varphi) = \int_\Omega \langle A \nabla f^i, \nabla f^i \rangle^{\frac{n-2}{2}} \langle A \nabla f^i, \nabla \varphi \rangle dx.$$

Since ∇f^i is the distributional gradient, a calculation shows (see [20, p. 379]) that

$$\langle A(x) \nabla f^i(x), \nabla f^i(x) \rangle^{\frac{n-2}{2}} A(x) \nabla f^i(x) = [\operatorname{adj} Df(x)]_i$$

where $[\operatorname{adj} Df(x)]_i$ denotes the i -th column of the adjoint matrix of $Df(x)$. Since $f \in W_{loc}^{1,n-1}(\Omega, \mathbb{R}^n)$ it is well known that (see [18, p. 256] or [20, p. 66])

$$\int_\Omega \langle [\operatorname{adj} Df(x)]_i, \nabla \varphi \rangle dx = 0 \quad i = 1, \dots, n,$$

for all $\varphi \in C_0^\infty(\Omega)$. Since A satisfies (6.2), by Theorem 4.5 we get the desired result. \square

Finally we end with an example of a mapping of finite distortion that is discontinuous on a set of measure zero whose inner distortion function belongs to $A_{n'} \cap RH_n$.

Example 6.3. Let $B = B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and consider

$$f(x) = \frac{x}{|x|} \exp(|x|^\varepsilon)$$

where $\varepsilon > 0$ is a small quantity to be chosen later. Notice that there is no way to define f so that it is continuous at the origin. Calculations show (see [20, chapter 6] for more details) that

$$|Df(x)| = |x|^{-1} \exp(|x|^\varepsilon) \quad \text{and} \quad J_f(x) = \varepsilon |x|^{\varepsilon-n} \exp(n|x|^\varepsilon).$$

We have that $f \in W^{1,p}(B, \mathbb{R}^n)$ for all $1 \leq p < n$ but $f \notin W^{1,n}(B, \mathbb{R}^n)$. Moreover,

$$K_O(x) = \varepsilon^{-1} |x|^{-\varepsilon} \quad \text{and} \quad K_I(x) = (\varepsilon^{-1} |x|^{-\varepsilon})^{n-1}.$$

If $\varepsilon > 0$ then neither K_I nor K_O belong to $ExpL$. However, if $\varepsilon < \frac{1}{n-1}$ then $K_I \in A_{n'} \cap RH_n$ and $MK_I(x) \sim |x|^{-\varepsilon(n-1)}$; thus f is discontinuous at $x = 0$, exactly where $MK_I = \infty$.

REFERENCES

- [1] K. Astala, T. Iwaniec, and G. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009.
- [2] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [3] M. Bramanti and L. Brandolini. Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs. *Rev. Mat. Iberoamericana*, 21(2):511–556, 2005.
- [4] L. Capogna, D. Danielli, and N. Garofalo. An embedding theorem and the Harnack inequality for nonlinear subelliptic equations. *Comm. Partial Differential Equations*, 18(9-10):1765–1794, 1993.
- [5] S Chanillo and R. L. Wheeden. Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions. *Amer. J. Math.*, 107(5):1191–1226, 1985.
- [6] S. Chanillo and R. L. Wheeden. Harnack’s inequality and mean-value inequalities for solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 11(10):1111–1134, 1986.
- [7] D. Cruz-Uribe, P. di Gironimo, and C. Sbordone. On the continuity of solutions to degenerate elliptic equations. *J. Diff. Eq.*, 250(6):2671–2686, 2011.
- [8] D. Danielli, N. Garofalo, and N. C. Phuc. Inequalities of Hardy-Sobolev type in Carnot-Carathéodory spaces. In *Sobolev spaces in mathematics. I*, volume 8 of *Int. Math. Ser. (N. Y.)*, pages 117–151. Springer, New York, 2009.
- [9] L. C. Evans. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations*, 45(3):356–373, 1982.
- [10] E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.
- [11] F. Ferrari. Harnack inequality for two-weight subelliptic p -Laplace operators. *Math. Nachr.*, 279(8):815–830, 2006.
- [12] B. Franchi, G. Lu, and R. L. Wheeden. Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. *Ann. Inst. Fourier (Grenoble)*, 45(2):577–604, 1995.
- [13] B. Franchi, G. Lu, and R. L. Wheeden. Weighted Poincaré inequalities for Hörmander vector fields and local regularity for a class of degenerate elliptic equations. *Potential Anal.*, 4(4):361–375, 1995. Potential theory and degenerate partial differential operators (Parma).
- [14] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.
- [15] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [16] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.
- [17] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.
- [18] J. Heinonen and P. Koskela. Sobolev mappings with integrable dilatations. *Arch. Rational Mech. Anal.*, 125(1):81–97, 1993.
- [19] T. Iwaniec, P. Koskela, and J. Onninen. Mappings of finite distortion: monotonicity and continuity. *Invent. Math.*, 144(3):507–531, 2001.
- [20] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.
- [21] R. Johnson and C. J. Neugebauer. Change of variable results for A_p - and reverse Hölder RH_r -classes. *Trans. Amer. Math. Soc.*, 328(2):639–666, 1991.

- [22] J. Kauhanen, P. Koskela, J. Malý, J. Onninen, and X. Zhong. Mappings of finite distortion: sharp Orlicz-conditions. *Rev. Mat. Iberoamericana*, 19(3):857–872, 2003.
- [23] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.
- [24] J. L. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858, 1983.
- [25] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
- [26] P. Lindqvist. *Notes on the p -Laplace equation*, volume 102 of *Report. University of Jyväskylä Department of Mathematics and Statistics*. University of Jyväskylä, Jyväskylä, 2006.
- [27] G. Lu. Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander’s condition and applications. *Rev. Mat. Iberoamericana*, 8(3):367–439, 1992.
- [28] J. J. Manfredi. Weakly monotone functions. *J. Geom. Anal.*, 4(3):393–402, 1994.
- [29] G. Modica. Quasiminima of some degenerate functionals. *Ann. Mat. Pura Appl. (4)*, 142:121–143 (1986), 1985.
- [30] D. D. Monticelli, S. Rodney, and R. L. Wheeden. Boundedness of weak solutions of degenerate quasilinear equations with rough coefficients. *preprint*, 2011.
- [31] A. Nagel, E. M. Stein, and S. Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.*, 155(1-2):103–147, 1985.
- [32] E. T. Sawyer and R. L. Wheeden. Degenerate Sobolev spaces and regularity of subelliptic equations. *Trans. Amer. Math. Soc.*, 362(4):1869–1906, 2010.
- [33] N. N. Ural’ceva. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 7:184–222, 1968.
- [34] S. K. Vodop’janov and V. M. Gol’dšteĭn. Quasiconformal mappings, and spaces of functions with first generalized derivatives. *Sibirsk. Mat. Ž.*, 17(3):515–531, 715, 1976.
- [35] X. Zhong. Discontinuous solutions of linear, degenerate elliptic equations. *J. Math. Pures Appl. (9)*, 90(1):31–41, 2008.

DEPARTMENT OF MATHEMATICS, TRINITY COLLEGE, HARTFORD, CT 06106, USA.

E-mail address: david.cruzuribe@trincoll.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, BOX 870350, 345 GORDON PALMER HALL.

E-mail address: kmoen@as.ua.edu

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, 138 CARDWELL HALL, MANHATTAN, KS 66506 USA.

E-mail address: vnaibo@math.ksu.edu